A limit theorem for the maximal interpoint distance of a random sample in the unit ball*

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Abstract

We prove a limit theorem for the the maximal interpoint distance (also called the diameter) for a sample of n i.i.d. points in the unit d-dimensional ball for $d \geq 2$. The exact form of the limit distribution and the required normalisation are derived using assumptions on the tail of the interpoint distance for two i.i.d. points. The results are specialised for the cases when the points have spherical symmetric distributions, in particular, are uniformly distributed in the whole ball and on its boundary.

Keywords: convex hull, extreme value, interpoint distance, Poisson process, random diameter, random polytope

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1 Introduction

Asymptotic behaviour of random polytopes formed by taking convex hulls of samples of i.i.d. points has been thoroughly investigated in the literature, see, e.g., [9, 18] for surveys of classical results and [17] for more recent studies. Consider a *random polytope* P_n obtained as the convex hull of n i.i.d. points $\xi_1, \dots, \xi_n$ sampled from the Euclidean space $\mathbb{R}^d$.

Most of results about random convex hulls are available in the planar case, i.e. for $d = 2$. The typical questions about random polytopes P_n concern the limit theorems for the geometric characteristics of P_n, for instance the area, the perimeter and the number of vertices of P_n, see [3, 8, 18]. Further important results concern the quantities that characterise the *worst case*

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approximation, notably the Hausdorff distance between K and P_n, see [4, 5]. It is well known [19] that the Hausdorff distance between two convex sets equals the uniform distance between their support functions defined on the unit sphere, i.e.

$$\rho_{\mathrm{H}}(P_n, K) = \sup_{u:\, \|u\|=1} (h(K,u) - h(P_n,u)) \,,$$

where $\|u\|$ is the Euclidean norm of $u \in \mathbb{R}^d$,

$$h(K,u) = \sup\{\langle u, x\rangle :\ x \in K\}$$

is the support function of K (and similar for P_n) and $\langle u, x\rangle$ is the scalar product in $\mathbb{R}^d$. For instance [5] shows that for uniformly distributed points $\rho_{\mathrm{H}}(P_n, K)$ is of order $\mathcal{O}((n^{-1}\log n)^{2/(d-1)})$ if K is sufficiently smooth.

The results on the *best case* approximation concern the behaviour of the infimum of the difference between $h(K,u)$ and $h(P_n,u)$. One of the few results in this direction states that if K is smooth, then $n(h(K,u) - h(P_n,u))$ (as a stochastic process indexed by u from the unit sphere $\mathbb{S}^{d-1}$) epi-converges in distribution to a certain process derived from the Poisson point process on $\mathbb{S}^{d-1} \times [0,\infty)$, see [15] and [14, Th. 5.3.34]. The epi-convergence implies the weak convergence of infima on each compact set. In particular, $n \inf_{u\in\mathbb{S}^{d-1}}(h(K,u) - h(P_n,u))$ converges in distribution to an exponentially distributed random variable, i.e. the best approximation error is of the order of n^{-1}. If the points are uniformly distributed in K, then this exponential random variable has the mean being the ratio of the volume of K and its surface area, see [14, Ex. 5.3.35]. Further results along these lines can be found in [20].

The best case approximation can be also studied by considering how fast the diameter of P_n, $\operatorname{diam} P_n$, approximates $\operatorname{diam} K$. By *diameter* we understand the maximum distance between any two points from the set. Note that $\operatorname{diam} K$ is not necessarily equal to the diameter of the smallest ball that contains K. This is the case, e.g. if K is a triangle.

A limit theorem for the diameter of P_n was proved in [2] for uniformly distributed points in a compact set K with unique longest chord (whose length is the diameter) and such that the boundary of K near the endpoints of this major axis is locally defined by regularly varying functions with indices strictly larger than 0.5. These assumptions are fairly restrictive and exclude a number of interesting smooth sets K, in particular balls and ellipsoids.

For K being the unit disk on the plane, [2] provides only bounds for the limit distribution, even without proving the existence of the limit. In

particular, [2, Th. 4] states that

$$\begin{aligned}1-\exp\left\{-\frac{4t^{5/2}}{3^{5/2}\pi}\right\} &\le \liminf_{n\to\infty}\mathbf{P}\{n^{4/5}(2-\operatorname{diam}P_n)\le t\}\\ &\le \limsup_{n\to\infty}\mathbf{P}\{n^{4/5}(2-\operatorname{diam}P_n)\le t\}\\ &\le 1-\exp\left\{-\frac{4t^{5/2}}{\pi}\right\}, \quad t>0\,. \qquad (1.1)\end{aligned}$$

In the classical theory of extreme values it is possible either to normalise the maximum of a random sample by dividing or multiplying its (possibly shifted or translated) maximum with normalising constants that grow to infinity. The first case corresponds to samples with possibly unbounded values, while the second one appears if samples with a finite right end-point of the distribution are considered. Quite similarly, in the extreme problems for random polytopes one can consider samples supported by the whole $\mathbb{R}^d$ or by a compact convex subset K in $\mathbb{R}^d$. In this paper we consider only the latter case. The limit theorems for the largest interpoint distances for samples from the whole $\mathbb{R}^d$ have been proved in [13] for the normally distributed samples and in [11] for more general spherically symmetric samples.

In this paper we state limit laws of the diameters of P_n, where P_n is the convex hull of a sample $\Xi_n = \{\xi_1, \dots, \xi_n\}$ of independent points distributed in the d-dimensional unit ball

$$B = \{x \in \mathbb{R}^d : \|x\| \le 1\}$$

according to some probability measure κ. The diameter of a set $F \subset \mathbb{R}^d$ is determined by its largest interpoint distance, i.e. by

$$\operatorname{diam}(F) = \sup_{x,y\in F}\|x-y\|\,,$$

and it is obvious that the diameter of F equals the diameter of its convex hull. In the special case when κ is the uniform distribution, the following result provides a considerable improvement of [2, Th. 4].

Theorem 1.1. *As $n\to\infty$, the diameter of the convex hull P_n of n independent points distributed uniformly on the d-dimensional unit ball B, $d \ge 2$, has limit distribution given by*

$$\mathbf{P}\{n^{\frac{4}{d+3}}(2-\operatorname{diam}P_n)\le t\} \to 1-\exp\left\{-\frac{2^d d\Gamma(\frac{d}{2}+1)}{\sqrt{\pi}(d+1)(d+3)\Gamma(\frac{d+1}{2})}t^{\frac{d+3}{2}}\right\},$$
$$t>0,$$

where $\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}ds$ denotes the Gamma function.

This theorem is proved by showing that the same limit distribution is shared by the diameter of a homogeneous Poisson process Π of constant intensity $\lambda = n/\mu_d(B)$ restricted on B, so that the total number of points in Π has mean n. See Section 3 for a more general de-Poissonisation argument, which implies that the diameter of a general binomial process with n points and of the corresponding Poisson process share the same limiting distributions (if the limit distribution exists).

The problem in dimension 1 is very easy to solve, see e.g. [6]. It is interesting to note that if all $\binom{n}{2}$ random distances $\|\xi_i - \xi_j\|$ are treated as an i.i.d. sequence with the common distribution determined by the length of the random chord in K, then the maximum of these distances has the same limit law as described in Theorem 1.1. This is explained by the fact that only different pairs of points contribute to $\operatorname{diam} P_n$, while the probability that a point has considerably large interpoint distances with two or more other points is negligible. This argument stems from [21] and was used in the proofs in [13] and [11]. Our proof relies on properties of the Poisson process with a subsequent application of a de-Poissonisation argument.

In Section 2 we establish the asymptotic behaviour of the diameter for a Poisson point process in B with growing intensity. The conditions on the intensity κ of the Poisson point process require certain asymptotic behaviour of the distance between two typical points of the process and a certain boundedness condition on κ. For instance, these conditions are fulfilled in the uniform case.

In Section 4 we investigate the asymptotic behaviour of the diameter of $\Pi_{n\kappa}$, where κ is a spherically symmetric distribution. Section 5 describes several examples, in particularly, where κ is the uniform measure on B and on $\mathbb{S}^{d-1}$, respectively. Further examples concern distributions which are not spherically symmetric.

The ball of radius r centered at the origin is denoted by B_r. By μ_d we denote the d-dimensional Lebesgue measure in $\mathbb{R}^d$. Furthermore, μ_{d-1} is the surface area measure on the unit sphere $\mathbb{S}^{d-1}$. By κ we understand a certain fixed probability measure on B and $\xi_1, \xi_2, \dots$ are i.i.d. points distributed according to κ.

For any set F in $\mathbb{R}^d$, $\check{F}$ denotes the reflected set $\{-x:\ x \in F\}$ and $\tilde{F}$ is the corresponding difference set

$$\tilde{F} = F + \check{F} = \{x - y:\ x, y \in F\}\,.$$

Finally, the letter Π_ν stands for the Poisson process on B of intensity measure ν, where we write shortly Π if no ambiguity occurs or the intensity

measure is immaterial. Note that $\Pi(F)$ denotes the number of points of a point process inside a set F, so that $\Pi(F)=0$ is equivalent to $\Pi\cap F=\emptyset$.

2 Diameters for Poisson processes

Consider a Poisson process $\Pi=\Pi_{n\kappa}$ in the unit ball B with the intensity measure proportional to a probability measure κ on B. Consider the convolution of κ with the reflected κ, i.e. the probability measure κ that determines the distribution of $\tilde{\xi}=\xi_1-\xi_2$ for i.i.d. ξ_1,ξ_2 distributed according to κ. Assume throughout that the support of $\tilde{\kappa}$ contains points with norms arbitrarily close to 2. In this case the diameter of $\Pi_{n\kappa}$ approaches 2 as $n\to\infty$. In this section we determine the asymptotic distribution of $2-\operatorname{diam}(\Pi_{n\kappa})$ as $n\to\infty$.

The distribution of the diameter of Π is closely related to the probability that the inner s-shell $B_2\setminus B_{2-s}$ of the ball of radius 2 contains no points of $\tilde{\Pi}=\Pi+\check{\Pi}$. Indeed

$$\mathbf{P}\{\operatorname{diam}\Pi\le 2-s\}=\mathbf{P}\{\tilde{\Pi}(B_2\setminus B_{2-s})=0\}\,,$$

and by the symmetry of $\tilde{\Pi}$,

$$\mathbf{P}\{\operatorname{diam}\Pi\le 2-s\}=\mathbf{P}\{\tilde{\Pi}((B_2\setminus B_{2-s})\cap H)=0\}\,, \tag{2.1}$$

where H is any halfspace bounded by a $(d-1)$-dimensional hyperplane passing through the origin.

For each $A\subset\mathbb{S}^{d-1}$ define

$$A_s=\{rx:\ x\in A,\ r\in[2-s,2]\}\,. \tag{2.2}$$

For each point $u\in\mathbb{S}^{d-1}$ define a cap of the unit ball of height $s\in(0,1)$ by

$$D_s(u)=B\cap\{x\in\mathbb{R}^d:\ \langle x,u\rangle\ge 1-s\}\,,$$

where $\langle x,u\rangle$ denotes the scalar product. For $A\subset\mathbb{S}^{d-1}$ define

$$D_s(A)=\cup_{u\in A}D_s(u)\,,\quad s\in(0,1)\,.$$

Then $D_s(A)$ and $D_s(\check{A})$ are subsets of $B\setminus B_{1-s}$ with the property that $x_1-x_2\in A_s$ for some $x_1,x_2\in B$ implies that x_1 belongs to $D_s(A)$ and x_2 to $D_s(\check{A})$.

Lemma 2.1. *For each $A\subset\mathbb{S}^{d-1}$, $s\in(0,1)$ and each $x_1\in B\setminus D_s(A)$ and $x_2\in B$, we have $x_1-x_2\notin A_s$.*

Proof. By definition of $D_s(u)$ and the fact that $\|x_2\| \le 1$, the inequality

$$\langle u, x_1 - x_2 \rangle = \langle u, x_1 \rangle + \langle u, -x_2 \rangle < 2 - s$$

holds for each $u \in A$. If $x_1 - x_2 \in A_s$, then $\|x_1 - x_2\| \ge 2 - s$ and $u_0 = (x_1 - x_2)\|x_1 - x_2\|^{-1} \in A$. Now write $2 - s \le \|x_1 - x_2\| = \langle u_0, x_1 - x_2 \rangle$, which is a contradiction to the first inequality, and hence the claim. $\square$

Lemma 2.2. *For each $s \in (0, 1)$ and $A \subset \mathbb{S}^{d-1}$, the set $D_s(A)$ lies inside the $\sqrt{2s}$-neighbourhood of A.*

Proof. Consider arbitrary $u \in A$. Since

$$\|x - u\|^2 = \|x\|^2 + \|u\|^2 - 2\langle x, u \rangle \le 2 - 2(1 - s) = 2s\,,$$

every point $x \in D_s(u)$ is located within distance at most $\sqrt{2s}$ from u. $\square$

The following lemma follows from Lemma 2.1 and the independence property of the Poisson process.

Lemma 2.3. *For any $A \subset \mathbb{S}^{d-1}$ and $s \in (0, 1)$,*

$$\mathbf{P}\{\tilde{\Pi} \cap A_s \ne \emptyset\} = \mathbf{P}\{\tilde{\Pi} \cap A_s \ne \emptyset, \Pi \cap D_s(A) \ne \emptyset, \check{\Pi} \cap D_s(A) \ne \emptyset\}\,. \quad (2.3)$$

If $A', A'' \subset \mathbb{S}^{d-1}$ and

$$D_s(A') \cap D_s(A'') = D_s(A') \cap D_s(\check{A}'') = \emptyset\,,$$

then the random variables $\tilde{\Pi}(A'_s)$ and $\tilde{\Pi}(A''_s)$ are independent.

The following lemma bounds $\mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap A_s \ne \emptyset\}$ using $\mathbf{P}\{\xi_1 - \xi_2 \in A_s\}$ for independent points ξ_1 and ξ_2 distributed according to κ.

Lemma 2.4. *For each $A \subset \mathbb{S}^{d-1}$ and $0 < s < 1$, we have*

$$\begin{aligned} n^2 e^{-n(a+\check{a})}\mathbf{P}\{\tilde{\xi} \in A_s\} &\le \mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap A_s \ne \emptyset\} \\ &\le n^2(1 + na\check{a}(a + \check{a}))\mathbf{P}\{\tilde{\xi} \in A_s\}\,, \end{aligned}$$

where $a = \kappa(D_s(A))$, $\check{a} = \kappa(D_s(\check{A}))$ and $\tilde{\xi} = \xi_1 - \xi_2$ for ξ_1 and ξ_2 being independent points distributed according to κ.

Proof. Let ζ_1 and ζ_2 be Poisson distributed with means na and $n\check{a}$ respectively, so that ζ_1 and ζ_2 represent the numbers of points of Π in $D_s(A)$ and $D_s(\check{A})$ respectively. First, (2.3) implies that

$$\begin{aligned} \mathbf{P}\{\tilde{\Pi} \cap A_s \ne \emptyset\} &= \mathbf{P}\{\tilde{\Pi}(A_s) \ge 1, \zeta_1 \ge 1, \zeta_2 \ge 1\} \\ &\ge \mathbf{P}\{\tilde{\Pi}(A_s) = 1, \zeta_1 = 1, \zeta_2 = 1\}\,. \end{aligned}$$

An upper bound follows from

$$\begin{aligned}\mathbf{P}\{\tilde{\Pi} \cap A_s \neq \emptyset\} &= \mathbf{P}\{\tilde{\Pi}(A_s) \geq 1, \zeta_1 \geq 1, \zeta_2 \geq 1\} \\ &\leq \mathbf{P}\{\tilde{\Pi}(A_s) = 1, \zeta_1 = 1, \zeta_2 = 1\} + I\,,\end{aligned}$$

where

$$I = \sum_{\substack{k_1, k_2 = 2, \\ \max(k_1, k_2) \geq 2}}^{\infty} \mathbf{P}\{\tilde{\Pi}(A_s) \geq 1, \zeta_1 = k_1, \zeta_2 = k_2\}\,.$$

The subadditivity of probability and the fact that ζ_1 and ζ_2 are independent immediately imply that

$$\mathbf{P}\{\tilde{\Pi}(A_s) \geq 1 | \zeta_1 = k_1, \zeta_2 = k_2\} \leq k_1 k_2 \mathbf{P}\{\xi_1 - \xi_2 \in A_s\}\,.$$

Thus,

$$\begin{aligned} I &\leq \mathbf{P}\{\xi_1 - \xi_2 \in A_s\} \left(\mathbf{E}(\zeta_1\zeta_2) - \mathbf{P}\{\zeta_1 = 1\}\mathbf{P}\{\zeta_2 = 1\}\right) \\ &= \mathbf{P}\{\xi_1 - \xi_2 \in A_s\}(n^2 a\breve{a} - n^2 a\breve{a} e^{-n(a+\breve{a})}) \\ &\leq \mathbf{P}\{\xi_1 - \xi_2 \in A_s\} n^3 a\breve{a}(a + \breve{a})\,.\end{aligned}$$

Now write

$$\begin{aligned}\mathbf{P}\{\tilde{\Pi}(A_s) = 1, \zeta_1 = 1, \zeta_2 = 1\} &= \mathbf{P}\{\tilde{\Pi}(A_s) = 1\ |\zeta_1 = 1, \zeta_2 = 1\} n^2 a\breve{a} e^{-n(a+\breve{a})} \\ &= \mathbf{P}\{\eta_1 - \eta_2 \in A_s\} n^2 a\breve{a} e^{-n(a+\breve{a})}\,,\end{aligned}$$

where η_1 and η_2 are independent points distributed according to the normalised measure κ restricted onto $D_s(A)$ and $D_s(\breve{A})$ respectively. Because of Lemma 2.1,

$$\mathbf{P}\{\eta_1 - \eta_2 \in A_s\} = \frac{1}{a\breve{a}} \mathbf{P}\{\xi_1 - \xi_2 \in A_s\}\,,$$

and the proof is complete. □

Let

$$C(u, r) = \{x \in \mathbb{S}^{d-1} : \|x - u\| \leq r\}\,, \quad u \in \mathbb{S}^{d-1},\ r > 0\,,$$

denote the spherical ball, so that

$$C_s(u, r) = \{rx : x \in C(u, r), r \in [2 - s, 2]\}$$

in accordance with (2.2).

Introduce the following assumption on the distribution of the difference $\tilde{\xi}$ between two independent points in B distributed according to κ. Assume that for a finite non-trivial measure σ on $\mathbb{S}^{d-1}$, some $\gamma > 0$ and $[\delta', \delta''] \subset (0, \frac{1}{2})$ we have

$$\lim_{s \downarrow 0} \frac{\mathbf{P}\{\tilde{\xi} \in C_s(u, z_s)\}}{s^\gamma \sigma(C(u, z_s))} = 1 \tag{2.4}$$

and

$$\lim_{s \downarrow 0} s^{-\gamma/2} \kappa(D_s(C(u, z_s))) = 0 \tag{2.5}$$

uniformly in $u \in \mathbb{S}^{d-1}$ and $z_s \in [s^{\delta'}, s^{\delta''}]$. If u does not belong to the support of σ, then the denominator in (2.4) equals zero for all sufficiently small s, and (2.4) is understood as the fact that the numerator also equals zero for all sufficiently small s. Since $\tilde{\xi}$ has a centrally symmetric distribution, the measure σ is necessarily centrally symmetric.

Lemma 2.5. *If (2.4) holds with $\gamma < d + 1$, κ is absolutely continuous on $B_1 \setminus B_{1-s}$ for some $s > 0$ and possesses there a bounded density, then (2.5) holds with*

$$\frac{\gamma - 2}{2(d-1)} < \delta' \leq \delta'' < \frac{1}{2} \,. \tag{2.6}$$

Proof. It suffices to show that, for any given $u \in \mathbb{S}^{d-1}$,

$$\lim_{s \downarrow 0} s^{-\gamma/2} \mu_d(D_s(C(u, s^\delta))) = 0 \,.$$

By Lemma 2.2, noticing that $\delta < \frac{1}{2}$, this would follow from

$$s^{-\gamma/2} \mu_{d-1}(C(u, 2s^\delta)) s \to 0 \quad \text{as } s \downarrow 0 \,.$$

The latter is the case, since $-\frac{1}{2}\gamma + \delta(d-1) + 1 > 0$ for all $\delta \in [\delta', \delta'']$. Finally, $\gamma < d + 1$ implies that $\frac{\gamma-2}{2(d-1)} < \frac{1}{2}$, so that (2.6) makes sense. ☐

In general, (2.5) is weaker than the boundedness of the density of κ with respect to the Lebesgue measure, which would, e.g., exclude the case where κ is supported by $\mathbb{S}^{d-1}$.

Lemma 2.6. *If (2.4) and (2.5) hold, then, for every measurable set $A \subset \mathbb{S}^{d-1}$ and $c > 0$,*

$$\lim_{s \downarrow 0} \mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap A_s \neq \emptyset\} \leq c^2 \sigma(A) \,,$$

where $n = cs^{-\gamma/2}$.

Proof. Cover A by spherical balls $C(u_i, s^\delta)$, $i = 1, \dots, m$, of diameter s^δ, where $\delta \in [\delta', \delta'']$. Then

$$\mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap A_s \neq \emptyset\} \leq \sum_{i=1}^{m} \mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap C_s(u_i, s^\delta) \neq \emptyset\} .$$

By the choice of n, Lemma 2.4 and (2.4),

$$\begin{aligned}\mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap C_s(u_i, s^\delta) \neq \emptyset\} &\leq c^2 s^{-\gamma} \mathbf{P}\{\tilde{\xi} \in C_s(u_i, s^\delta)\}(1 + n a_i \check{a}_i) \\ &\leq c^2(1+\varepsilon)\sigma(C(u_i, s^\delta))(1 + n a_i \check{a}_i)\end{aligned}$$

for any $\varepsilon > 0$ and all sufficiently small s, where $a_i = \kappa(D_s(C(u_i, s^\delta)))$ and $\check{a}_i = \kappa(D_s(C(-u_i, s^\delta)))$. Condition (2.5) implies that $n a_i \check{a}_i \to 0$ as $s \downarrow 0$. Therefore,

$$\mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap C_s(u_i, s^\delta) \neq \emptyset\} \leq c^2(1+\varepsilon)^2 \sigma(C(u_i, s^\delta))$$

for all sufficiently small s. Thus,

$$\mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap A_s \neq \emptyset\} \leq c^2(1+\varepsilon)^2 \sum_{i=1}^{m} \sigma(C(u_i, s^\delta))$$

for all sufficiently small s. The statement is proven by taking infimum in the right-hand side over all possible ball-coverings of A. $\square$

In the following we need the following assumption on σ:

(S) σ is a measure on $\mathbb{S}^{d-1}$ with finite total mass σ_0 such that

$$\sigma(A) \leq f(\mu_{d-1}(A)) \tag{2.7}$$

for all measurable $A \subset \mathbb{S}^{d-1}$ with a function f such that $f(x) \to 0$ as $x \downarrow 0$.

It is easy to see that (2.7) holds if σ is absolutely continuous with respect to μ_{d-1} and has a bounded density. An atomic σ clearly violates **(S)**.

Theorem 2.7. *Assume that (2.4) and (2.5) hold with $\delta' < \delta''$ and a σ that satisfies **(S)**. Then*

$$\lim_{n\to\infty} \mathbf{P}\{n^{2/\gamma}(2 - \operatorname{diam}(\Pi_{n\kappa})) \leq t\} = 1 - e^{-\frac{1}{2}t^\gamma \sigma_0} , \quad t \geq 0 , \tag{2.8}$$

where $\sigma_0 = \sigma(\mathbb{S}^{d-1})$.

Proof. Let $\mathbb{S}_+^{d-1}$ denote the half-sphere, obtained by intersection $\mathbb{S}^{d-1}$ with any fixed half-space H, for instance given by (2.1). Fix any $\varepsilon > 0$ and consider disjoint spherical balls $C(x_i, s^{\delta_i})$, $i = 1, \dots, m$, where $x_i \in \mathbb{S}_+^{d-1}$ and $\delta_i \in [\delta', \delta'']$. Since these spherical balls are constructed using varying scales of s, it is possible to pack them arbitrarily dense as $s \downarrow 0$, i.e. assume that the Lebesgue measure of the uncovered part is smaller than ε.

Define the spherical balls

$$A^{(i)} = C(x_i, s^{\delta_i} - \sqrt{2s}), \quad i = 1, \dots, m.$$

Since $\sqrt{2s} \le s^{\delta_i}$ for all sufficiently small s, Lemma 2.2 implies that $D_s(A^{(i)})$, $i = 1, \dots, m$, are pairwise disjoint for all sufficiently small s. By Lemma 2.3, the random variables $\tilde{\Pi}(A_s^{(i)})$, $i = 1, \dots, m$, are independent.

Denote

$$\Delta(s) = \mathbb{S}_+^{d-1} \setminus (A^{(1)} \cup \dots \cup A^{(m)}) \tag{2.9}$$

to be the uncovered part of $\mathbb{S}_+^{d-1}$ left by the $A^{(i)}$'s. The Lebesgue measure of $\Delta(s)$ equals the sum of the μ_{d-1}-measure of the part left uncovered by $C(x_i, s^{\delta_i})$, $i = 1, \dots, m$, and the sum of the measures of $C(x_i, s^{\delta_i}) \setminus A^{(i)}$. Thus

$$\begin{aligned}
\mu_{d-1}(\Delta(s)) &\le \varepsilon + \sum_{i=1}^{m} c_1 (d-2) s^{\delta_i (d-2)} q \sqrt{2s} \\
&\le \varepsilon + \sum_{i=1}^{m} c_1 s^{\delta_i (d-1)} \sqrt{2s} s^{-\delta'} \\
&\le \varepsilon + c_2 \sqrt{2s} s^{-\delta'} \le 2\varepsilon
\end{aligned}$$

for all sufficiently small s, where c_1 and c_2 are positive constants. Since the chosen points $x_1, \dots, x_m$ do not include at most ε of the atomic part of σ, condition **(S)** implies that $\sigma(\Delta(s))$ is smaller than $\varepsilon + f(2\varepsilon)$ for all sufficiently small s. In turn, $\varepsilon + f(\varepsilon)$ can be made smaller than any given $\varepsilon' > 0$. By Lemma 2.6,

$$\lim_{s \downarrow 0} \mathbf{P}\{\tilde{\Pi} \cap \Delta_s(s) \ne \emptyset\} \le t^2 \varepsilon'. \tag{2.10}$$

For any fixed $t > 0$ consider the Poisson process Π with intensity measure $n\kappa$ with $n = ts^{-\gamma/2}$ for a fixed t. Then

$$\mathbf{P}\{\mathrm{diam}(\Pi) \le 2 - s\} = \mathbf{P}\{\tilde{\Pi} \cap A_s^{(i)} = \emptyset,\ i = 1, \dots, m,\ \tilde{\Pi} \cap \Delta_s(s) = \emptyset\}.$$

By the independence of $\tilde{\Pi}(A_s^{(i)})$, $i = 1, \dots, m$,

$$I \le \mathbf{P}\{2 - \mathrm{diam}(\Pi) \le s\} \le I + \mathbf{P}\{\tilde{\Pi} \cap \Delta_s(s) \ne \emptyset\},$$

where

$$I = 1 - \prod_{i=1}^{m} \mathbf{P}\{\tilde{\Pi}(A_s^{(i)}) = 0\}\,.$$

By Lemma 2.4,

$$\begin{aligned}\prod_{i=1}^{m}(1 - n^2(1 + y_1(s))\mathbf{P}\{\tilde{\xi} \in A_s^{(i)}\}) &\le \prod_{i=1}^{m} \mathbf{P}\{\tilde{\Pi}(A_s^{(i)}) = 0\} \\ &\le \prod_{i=1}^{m}(1 - n^2 e^{-2y_2(s)}\mathbf{P}\{\tilde{\xi} \in A_s^{(i)}\})\,,\end{aligned}$$

where

$$\begin{aligned} y_1(s) &= \max_{i=1,\dots,m} n\kappa(D_s(A^{(i)}))\kappa(D_s(\check{A}^{(i)}))\,, \\ y_2(s) &= \max_{i=1,\dots,m} n\kappa(D_s(A^{(i)}))\,. \end{aligned}$$

By (2.5), $y_2(s)$ (and thereupon also $y_1(s)$) converge to zero as $s \downarrow 0$ for $n = ts^{-\gamma/2}$. By (2.4) with $z_s = s^{\delta_i} - \sqrt{2s}$,

$$\frac{n^2 \mathbf{P}\{\tilde{\xi} \in A_s^{(i)}\}}{\sigma(A^{(i)})} \to t^\gamma \quad \text{as } s \downarrow 0\,.$$

Since $y_1(s) \to 0$ and $y_2(s) \to 0$,

$$\lim_{s\downarrow 0} \prod_{i=1}^{m}(1 - n^2(1 + y_1(s))\mathbf{P}\{\tilde{\xi} \in A_s^{(i)}\}) = \lim_{s\downarrow 0} \prod_{i=1}^{m}(1 - t^\gamma \sigma(A^{(i)}))\,,$$

and

$$\lim_{s\downarrow 0} \prod_{i=1}^{m}(1 - n^2 e^{-2y_2(s)}\mathbf{P}\{\tilde{\xi} \in A_s^{(i)}\}) = \lim_{s\downarrow 0} \prod_{i=1}^{m}(1 - t^\gamma \sigma(A^{(i)}))\,.$$

By taking logarithm, and using the inequality $|\log(1 + x) - x| \le |x|^2$ for $|x| < 1$, we see that

$$\begin{aligned}\lim_{s\downarrow 0} \prod_{i=1}^{m}(1 - t^\gamma \sigma(A^{(i)})) &= \exp\left\{-t^\gamma \lim_{s\downarrow 0} \sum_{i=1}^{m} \sigma(A^{(i)})\right\} \\ &= \exp\{-\frac{1}{2}t^\gamma \sigma_0\}\,.\end{aligned} \tag{2.11}$$

For this, note that σ is necessarily symmetric, so that $\sigma(\mathbb{S}_+^{d-1}) = \sigma_0/2$. Finally, (2.8) is obtained from (2.10) and the choice of $n = (t/s)^{-\gamma/2}$. $\square$

Instead of imposing **(S)** it is possible to request that for every $s > 0$ there exists a covering of $\mathbb{S}^{d-1}$ by spherical balls $C(x_i, s^{\delta_i})$ of radii s^{δ_i} with $\delta_i \in [\delta', \delta''] \subset (0, \frac{1}{2})$ such that $\sigma(\Delta(s)) \to 0$ as $s \downarrow 0$, where $\Delta(s)$ is given by (2.9). Since this condition always holds in dimension $d = 2$ with $\delta' = \delta''$, we obtain the following result for interpoint distances in the unit disk.

Theorem 2.8. *Assume that (2.4) and (2.5) hold with any fixed $\delta \in (0, \frac{1}{2})$ uniformly over $u \in \mathbb{S}^1$. Then (2.8) holds.*

Instead of imposing (2.4) and (2.5), it is possible to deduce the limiting distribution in (2.8) using a direct assumption on the asymptotic distribution of $\tilde{\Pi}_{n\kappa}$.

Theorem 2.9. *Assume that, for $[\delta', \delta''] \subset (0, \frac{1}{2})$ with $\delta' < \delta''$,*

$$\lim_{s \downarrow 0} \frac{\mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap C_s(u, z_s) \neq \emptyset\}}{\sigma(C(u, z_s))} = g(t, u) \tag{2.12}$$

*uniformly in $u \in \mathbb{S}^{d-1}$ and $z_s \in [s^{\delta'}, s^{\delta''}]$, where $n = ts^{-\gamma/2}$ and σ s a finite measure on $\mathbb{S}^{d-1}$. If the non-atomic part σ' of σ satisfies **(S)**, then*

$$\begin{aligned} &\lim_{n \to \infty} \mathbf{P}\{n^{2/\gamma}(2 - \mathrm{diam}(\Pi_{n\kappa})) \leq t\} \\ &= 1 - \exp\left\{-\frac{1}{2} \int_{\mathbb{S}^{d-1}} g(t, u) \sigma'(du)\right\} \prod_{\substack{x_i \in \mathbb{S}^{d-1} \\ \sigma(\{x_i\}) > 0}} \Big(1 - g(t, x_i)\sigma(\{x_i\})\Big)^{\frac{1}{2}} \end{aligned} \tag{2.13}$$

for all $t \geq 0$.

Proof. For the proof we use the same sets partition and the sets $A^{(i)}$ as in the proof of Theorem 2.7. If σ has an atomic part, choose the points $x_1, \ldots, x_m$ in such a way that they have so many atoms of σ among them that the total σ-content of the remaining atoms is less than ε.

In the remainder of the proof we need to split the product in the left-hand side of (2.11) into the factors that correspond to the non-atomic and the atomic parts of σ. Notice that Lemma 2.4 is no longer needed to derive the asymptotics for $\mathbf{P}\{\tilde{\Pi}(A_s^{(i)}) = 0\}$ from the distribution of $\tilde{\xi}$. The square root of the product appears because we need to count only atoms from the half-sphere. Alternatively it is possible to take the produst only over $x_i \in \mathbb{S}_+^{d-1}$. $\square$

The cases when σ has atoms often appear if κ is the (say uniform distribution) supported by a subset K of B and such that K is sufficiently "sharply

pointed" near the points where its diameter is achieved. The typical example of such K is a segment, see 5.5. Other examples correspond to sets that satisfy the conditions imposed in [2].

3 De-Poissonisation

Let Π be the Poisson process with intensity measure $n\kappa$. Given $\Pi(K) = n$, the distribution of Π coincides with the distribution of $\Xi_n = \{\xi_1, \dots, \xi_n\}$ being the binomial process on K that consists of i.i.d. points sampled from κ. In the other direction, the distribution of Π coincides with the distribution of Ξ_N, where N is the Poisson random variable of mean n independent of the i.i.d. points ξ_i's distributed according to κ. This simple relationship makes it possible to use the de-Poissonisation technique [16] in order to obtain the limit theorem for functionals of P_n being the convex hull of Ξ_n. The key issue that simplifies our proofs is the monotonicity of the diameter functional. Indeed, the diameter of Ξ_n is stochastically greater than the diameter of Ξ_m for $n > m$. Another useful tool is provided by the following lemma from [16, p. 18].

Lemma 3.1. *Let N be a Poisson random variable with mean λ. For every $\gamma > 0$ there exists a constant $\lambda_1 = \lambda_1(\gamma) \geq 0$ such that*

$$\mathbf{P}\{|N - \lambda| \geq \frac{1}{2}\lambda^{\frac{1}{2}+\gamma}\} \leq 2\exp\{-\frac{1}{9}\lambda^{2\gamma}\}$$

for all $\lambda > \lambda_1$.

Theorem 3.2. *Let $\Psi : \mathcal{N} \to \mathbb{R}$ be a monotonic functional defined on the space $\mathcal{N}$ of finite subsets of $\mathbb{R}^d$. Furthermore, let $\Pi_{n\kappa}$ be a Poisson process with intensity measure $n\kappa$ where κ is a probability measure on $\mathbb{R}^d$. If, for some $\alpha > 0$, the random variable $n^\alpha \Psi(\Pi_{n\kappa})$ converges in distribution to a random variable with cumulative distribution function F, then $n^\alpha \Psi(\Xi_n)$ also weakly converges to F, where Ξ_n is a binomial process of n i.i.d. points with common distribution κ.*

Proof. Without loss of generality assume that Ψ is non-decreasing. Define $\gamma = \frac{1}{2} - \beta$ and $\varepsilon_n = n^{-\beta}$ for some $\beta \in (0, \frac{1}{2})$. By Lemma 3.1 and the monotonicity of Ψ,

$$\begin{aligned}\mathbf{P}\{\Psi(\Pi_{n\kappa}) \leq s\} &\leq \mathbf{P}\{\Psi(\Pi_{n\kappa}) \leq s,\ |N - n| \leq n\varepsilon_n\} + \mathbf{P}\{|N - n| > n\varepsilon_n\} \\ &\leq \mathbf{P}\{\Psi(\Xi_{n(1-\varepsilon_n)}) \leq s\} + 2\exp\{-\frac{1}{9}(2n)^{2\gamma}\}\,.\end{aligned}$$

for sufficiently large n. Therefore, for every continuity point t of F,

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{P}\{\Psi(\Xi_n)n^\alpha \le t\} &= \lim_{n\to\infty} \mathbf{P}\{\Psi(\xi_{n(1-\varepsilon_n)})(n(1-\varepsilon_n))^\alpha \le t\} \\
&\ge \lim_{n\to\infty} \mathbf{P}\{\Psi(\Xi_{n(1-\varepsilon_n)})n^\alpha \le t\} \\
&\ge \lim_{n\to\infty} \mathbf{P}\{\Psi(\Pi_{n\kappa})n^\alpha \le t\} - 2\exp\{-\frac{1}{9}(2n)^{2\gamma}\} \\
&= \lim_{n\to\infty} \mathbf{P}\{\Psi(\Pi_{n\kappa})n^\alpha \le t\} \\
&= F(t)\,.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathbf{P}\{\Psi(\Pi) \le s\} &\ge \mathbf{P}\{\Psi(\Xi_{n(1+\varepsilon_n)}) \le s\}\mathbf{P}\{|N-n| \le n\varepsilon_n\} \\
&\ge \mathbf{P}\{\Psi(\Xi_{n(1+\varepsilon_n)}) \le s\} - 2\exp\{-\frac{1}{9}(2n)^{-2\gamma}\}\,,
\end{aligned}$$

so that

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{P}\{\Psi(\Xi_n)n^\alpha \le t\} &= \lim_{n\to\infty} \mathbf{P}\{\Psi(\xi_{n(1+\varepsilon_n)})(n(1+\varepsilon_n))^\alpha \le t\} \\
&\le \lim_{n\to\infty} \mathbf{P}\{\Psi(\Xi_{n(1+\varepsilon_n)})n^\alpha \le t\} \\
&\le \lim_{n\to\infty} \mathbf{P}\{\Psi(\Pi_{n\kappa})n^\alpha \le t\} - 2\exp\{-\frac{1}{9}(2n)^{2\gamma}\} \\
&= \lim_{n\to\infty} \mathbf{P}\{\Psi(\Pi_{n\kappa})n^\alpha \le t\} \\
&= F(t).
\end{aligned}$$

□

In particular, Theorem 3.2 is applicable for the functional $\Psi(\Xi_n) = 2 - \mathrm{diam}(\Xi_n)$, so that all results available for diameters of Poisson processes can be immediately reformulated for binomial processes.

4 Spherically symmetric distributions

Let $\xi_1, \dots, \xi_n$ be independent points distributed according to a spherically symmetric (also called "isotropic") density κ restricted on B. Spherically symmetric distributions are closed with respect to convolution, so that $\tilde{\xi} = \xi_1 - \xi_2$ is spherically symmetric too. Therefore, $\|\tilde{\xi}\|$ and $\tilde{\xi}/\|\tilde{\xi}\|$ are independent, see e.g. [7]. Then, for any measurable $A \subset \mathbb{S}^{d-1}$

$$\mathbf{P}\{\tilde{\xi} \in A_s\} = \mathbf{P}\{\|\tilde{\xi}\| \ge 2 - s\}\frac{\mu_{d-1}(A)}{\mu_{d-1}(\mathbb{S}^{d-1})}\,.$$

Therefore (2.4) is fulfilled if, for some $\gamma > 0$,

$$\lim_{s\to 0} \mathbf{P}\{\|\tilde{\xi}\| \geq 2-s\} s^{-\gamma} = \sigma_0 \in (0,\infty)\,, \tag{4.1}$$

where the limit σ_0 then becomes the total mass of σ, so that σ is the surface area measure on $\mathbb{S}^{d-1}$ normalised to have the total mass σ_0.

Furthermore, (2.5) holds if

$$\lim_{s\downarrow 0} s^{\delta(d-1)-\gamma/2} \mathbf{P}\{\eta \leq s\} = 0\,, \tag{4.2}$$

where $\eta = 1 - \|\xi_1\|$.

Lemma 4.1. *If η_1 and η_2 are i.i.d. distributed as $1-\|\xi_1\|$ and $\zeta = \eta_1 + \eta_2$, then*

$$\lim_{s\downarrow 0} \frac{\mathbf{P}\{\|\tilde{\xi}\| \geq 2-s\}}{\mathbf{E}((s-\zeta)^{(d-1)/2}\mathbf{1}_{\zeta\leq s})} = \frac{2^{d-1}\Gamma(\frac{d}{2})}{(d-1)\pi^{\frac{1}{2}}\Gamma(\frac{d-1}{2})}\,. \tag{4.3}$$

Proof. By the cosine theorem and the fact that $\tilde{\xi}$ has the same distribution as $\xi_1 + \xi_2$, we write

$$\mathbf{P}\{\|\tilde{\xi}\| \geq 2-s\} = \mathbf{P}\{\|\xi_1\|^2 + \|\xi_2\|^2 + 2\|\xi_1\|\|\xi_2\|\cos\beta \geq (2-s)^2\},$$

where β denotes the angle between ξ_1 and ξ_2. Hence,

$$\mathbf{P}\{\|\tilde{\xi}\| \geq 2-s\} = \mathbf{P}\{\cos\beta \geq 1-q\}\,,$$

where

$$q = \frac{(2-\zeta)^2 - (2-s)^2}{2\|\xi_1\|\|\xi_2\|}\,.$$

If $q \geq 0$ (i.e. $\zeta \leq s$)

$$\mathbf{P}\{\|\tilde{\xi}\| \geq 2-s\} = \frac{1}{2}\mathbf{P}\{\cos^2\beta \geq (1-q)^2, \zeta \leq s\} = \frac{1}{2}\mathbf{E}\Big(\int_{(1-q)^2}^{1} f(t)dt\mathbf{1}_{\zeta\leq s}\Big)\,,$$

where the probability density function

$$f(t) = \frac{\Gamma(\frac{d}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{d-1}{2})} t^{-\frac{1}{2}}(1-t)^{\frac{d-3}{2}}\,, \quad t \in [0,1]\,,$$

of $\cos^2\beta$ corresponds to the Beta-distribution with parameters $\frac{1}{2}$ and $(d-1)/2$, see [13, Prop. 2]. Substituting $x = 1-t$ leads to

$$\mathbf{P}\{\|\tilde{\xi}\| \geq 2-s\} = c_1\mathbf{E}\Big(\int_0^{2q(1-\frac{q}{2})} (1-x)^{-\frac{1}{2}} x^{\frac{d-1}{2}-1} dx\mathbf{1}_{\zeta\leq s}\Big)\,,$$

where

$$c_1 = \frac{1}{2}\frac{\Gamma(\frac{d}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{d-1}{2})}\,.$$

The inequality

$$1 \le (1-x)^{-\frac{1}{2}} \le (1-q)^{-1} \le ((1-s)^2 - 2s)^{-1}$$

leads to the bounds

$$c_1\mathbf{E}(I\mathbf{1}_{\zeta\le s}) \le \mathbf{P}\{\|\tilde{\xi}\| \ge 2-s\} \le c_1((1-s)^2-2s)^{-1}\mathbf{E}(I\mathbf{1}_{\zeta\le s})\,, \qquad (4.4)$$

where

$$I = \int_0^{2q(1-\frac{q}{2})} x^{\frac{d-1}{2}-1}dx = \frac{2}{d-1}(2q)^{\frac{d-1}{2}}(1-\frac{q}{2})^{\frac{d-1}{2}}.$$

By the fact that

$$1 \le (\|\xi_1\|\|\xi_2\|)^{-1} \le (1-s)^{-2}$$

and

$$1 - \frac{s}{(1-s)^2} \le 1 - \frac{q}{2} \le 1\,,$$

we further get the bounds

$$\frac{2}{d-1}((2-\zeta)^2-(2-s)^2)^{\frac{d-1}{2}}\left(1-\frac{s}{(1-s)^2}\right)^{\frac{d-1}{2}} \le I$$
$$\le \frac{2}{d-1}((2-\zeta)^2-(2-s)^2)^{\frac{d-1}{2}}(1-s)^{-(d-1)}\,.$$

Since

$$(4-2s)(s-\zeta) \le (2-\zeta)^2-(2-s)^2 \le 4(s-\zeta)\,,$$

the following bounds for I hold

$$\frac{2^d}{d-1}(s-\zeta)^{\frac{d-1}{2}}\left(1-\frac{s}{(1-s)^2}\right)^{\frac{d-1}{2}}(1-\frac{s}{2})^{\frac{d-1}{2}} \le I$$
$$\le \frac{2^d}{d-1}(s-\zeta)^{\frac{d-1}{2}}(1-s)^{-(d-1)}\,.$$

Plugging these bounds in (4.4) yields the result. $\square$

The following result settles the case when the density of η is equivalent to a power function for small arguments.

Theorem 4.2. *Assume that $d \geq 2$ and for some $\alpha \geq 0$ the cumulative distribution function $F(x) = \mathbf{P}\{\eta \leq x\}$ of η satisfies*

$$\lim_{s \downarrow 0} s^{-\alpha} F(s) = a \in (0, \infty)\,. \tag{4.5}$$

Then

$$\lim_{n\to\infty} \mathbf{P}\{n^{2/\gamma}(2 - \operatorname{diam}(\Pi_{n\kappa})) \leq t\} = 1 - e^{-\frac{1}{2}t^\gamma \sigma_0}\,, \quad t \geq 0\,, \tag{4.6}$$

where $\gamma = \frac{1}{2}(d-1) + 2\alpha$ and

$$\sigma_0 = a^2 c \begin{cases} 1\,, & F(0) > 0\,, \\ \alpha^2 \Gamma(\alpha)^2 \frac{\Gamma(\frac{1}{2}(d+1))}{\Gamma(2\alpha + \frac{1}{2}(d+1))}\,, & F(0) = 0\,, \end{cases}$$

with c given by the right-hand side of (4.3).

Proof. The integration by parts leads to

$$\begin{aligned} \mathbf{E}((s-\zeta)^{(d-1)/2}\mathbf{1}_{\zeta \leq s}) &= F(0)^2 s^{(d-1)/2} \\ &+ \frac{(d-1)(d-3)}{4} \int_0^s \int_0^{s-x_1} F(x_1)F(x_2)(s - x_1 - x_2)^{(d-5)/2} dx_1 dx_2\,. \end{aligned}$$

If $F(0) > 0$, then (4.5) implies that $\alpha = 0$, so that (4.1) holds with $\gamma = \frac{1}{2}(d-1)$ and $\sigma_0 = F(0)^2 c = a^2 c$ by Lemma 4.1.

If $F(0) = 0$, then (4.5) yields that $\mathbf{E}((s-\zeta)^{(d-1)/2}\mathbf{1}_{\zeta \leq s})$ is equivalent as $s \downarrow 0$ to

$$\begin{aligned} s^\gamma a^2 \frac{(d-1)(d-3)}{4} & \int_0^1 \int_0^{1-t_1} t_1^\alpha t_2^\alpha (1 - t_1 - t_2)^{(d-5)/2} dt_1 dt_2 \\ &= s^\gamma a^2 \frac{(d-1)(d-3)}{4} \mathrm{B}(\alpha+1, \alpha + \frac{d-1}{2}) \mathrm{B}(\alpha+1, \frac{d-3}{2}) \\ &= s^\gamma a^2 \alpha^2 \Gamma(\alpha)^2 \frac{\Gamma(\frac{1}{2}(d+1))}{\Gamma(2\alpha + \frac{1}{2}(d+1))} \end{aligned}$$

with $\gamma = \frac{1}{2}(d-1) + 2\alpha$. Finally, (4.1) follows from Lemma 4.1. It remains to show that (4.2) holds, i.e.

$$\delta(d-1) - \frac{1}{2}\gamma + \alpha > 0\,.$$

Using the expression for γ, it suffices to note that $\delta(d-1) - \frac{1}{4}(d-1) > 0$ if $\delta \in (\frac{1}{4}, \frac{1}{2})$, so it is possible to choose $[\delta', \delta''] \subset (\frac{1}{4}, \frac{1}{2})$. $\square$

It should be noted that (4.5) can be replaced by the requirement that F is regular varying at zero. However, in this case the constants involved in the formula for σ_0 are given by the integrals of the slowly varying part of F.

Using similar arguments, it is possible to check (2.4) and (2.5) if $\xi = \eta u$ for independent η and u, where η distributed on $[0,1]$ and u is distributed on $\mathbb{S}^{d-1}$.

5 Examples

5.1 Uniformly distributed points in the ball

Consider the case of random points uniformly distributed in B.

Theorem 5.1. *As $n \to \infty$, the diameter of the convex hull of a homogeneous Poisson process Π_λ with intensity $\lambda = n/\mu_d(B)$ restricted on the d-dimensional unit ball, $d \geq 2$, has limit distribution*

$$\mathbf{P}\{n^{\frac{4}{d+3}}(2 - \operatorname{diam}\Pi_\lambda) \leq t\} \to 1 - \exp\left\{-\frac{1}{2}\, c\, t^{\frac{d+3}{2}}\right\}, \quad t > 0\,,$$

where

$$c = \frac{2^{d+1} d\Gamma(\frac{d}{2}+1)}{\sqrt{\pi}(d+1)(d+3)\Gamma(\frac{d+1}{2})}\,. \tag{5.1}$$

Proof. The tail behaviour of $\|\xi_1\|$ is determined by

$$\mathbf{P}\{\|\xi_1\| \geq 1 - s\} = 1 - \frac{\mu_d(B_{1-s})}{\mu_d(B)} = 1 - (1-s)^d\,,$$

so that Theorem 4.2 is applicable with $\alpha = 1$ and $a = d$. □

By the de-Poissonisation argument, Theorem 5.1 yields Theorem 1.1. Note that in case $d = 2$ the constant c equals $16/(15\pi)$, which also corresponds to the bounds given in (1.1). The tail behaviour of $\|\tilde{\xi}\|$ can also be obtained from the explicit formula for the distribution of the length of a random chord in the unit ball, see [12, 2.48].

5.2 Uniformly distributed points on the sphere

Another example of a spherically symmetric distribution is given by the uniform distribution on $\mathbb{S}^{d-1}$, i.e. if

$$\kappa(A) = \mu_{d-1}(A)/\mu_{d-1}(\mathbb{S}^{d-1})$$

for all measurable $A \subset \mathbb{S}^{d-1}$. The following result follows from Theorem 4.2 in case $a = F(0) = 1$ and $\alpha = 0$.

Theorem 5.2. *If Π is the homogeneous Poisson process on $\mathbb{S}^{d-1}$ with the total intensity n, then for any $d \geq 2$*

$$\lim_{n\to\infty} \mathbf{P}\{n^{\frac{4}{d-1}}(2 - \mathrm{diam}(\Pi)) \leq t\} = 1 - \exp\left\{-\frac{1}{2}\, c\, t^{\frac{d-1}{2}}\right\}, \quad t > 0,$$

where

$$c = \frac{2^{d-1}\Gamma(\frac{d}{2})}{(d-1)\pi^{\frac{1}{2}}\Gamma(\frac{d-1}{2})}. \tag{5.2}$$

Alternatively, the tail behaviour of $\|\tilde{\xi}\|$ may be derived from the explicit formula for the distribution of the distance between two uniform points on the unit sphere, see [1].

Similarly, it is possible to obtain limit theorems for a spherically symmetric ξ in case the norm $\|\xi\|$ has a rather general distribution which is regular varying near its right end-point being 1.

5.3 Distribution in spherical sectors

This section provides a simple example, where κ is not spherically symmetric. Consider some spherically symmetric measure κ' which satisfies (4.1) and (4.2) for some c and γ, and a spherical sector L defined by

$$L = \{tx : x \in A, t \in [-1, 1]\}$$

for some fixed $r > 0$, where A is a spherically convex subset of the unit sphere. If $L \neq B$, then

$$\kappa(S) = \kappa'(S \cap L)/\kappa'(L)$$

defines a not spherically symmetric measure for all measurable $S \subset B$. Denote by ξ_1 and ξ_2 two independent points sampled from κ and by ξ_1', ξ_2' two independent points distributed according to κ', respectively. By the construction of κ and the spherical symmetry of κ' we can write

$$\mathbf{P}\{\tilde{\xi} \in C_s(u, r)\} = \mathbf{P}\{\|\xi_1' - \xi_2'\| \geq 2 - s\}\frac{\mu_{d-1}(C(u, r)) \cap A)}{\mu_{d-1}(A)} \tag{5.3}$$

for all spherical balls $C_s(u, r)$. For all measurable $F \subset \mathbb{S}^{d-1}$ define

$$\sigma(F) = c\,\frac{\mu_{d-1}(F \cap A)}{\mu_{d-1}(\mathbb{S}^{d-1})}.$$

It is easy to verify that σ satisfies condition (2.4), since by (5.3)

$$\lim_{s\downarrow 0}\sup_{u\in\operatorname{supp}\sigma}\left|\frac{\mathbf{P}\{\tilde{\xi}\in C_s(u,r)\}}{s^\gamma\sigma(C(u,r))}-1\right|=0\,.$$

Since

$$\kappa(D_s(C_s(u,s^\delta)))=\kappa'(D_s(C_s(u,s^\delta))\cap L)/\kappa'(L)\,,$$

condition (2.5) is fulfilled and Theorem 2.7 holds with

$$\sigma_0=c\frac{\mu_{d-1}(A)}{\mu_{d-1}(\mathbb{S}^{d-1})}\,.$$

5.4 Non-uniform angular distributions

Assume that ξ is distributed on the boundary of the unit circle in $\mathbb{R}^2$ according to some not necessarily symmetrical probability measure κ, which can be then considered a measure on $[0,2\pi)$. If ξ_1 and ξ_2 are distributed on $[0,2\pi)$ according to κ, then

$$\begin{aligned}\mathbf{P}\{1-\cos(\xi_1-\xi_2)\le 2s(1-s/2)(1-s)^{-2},\ |\xi_1+\xi_2-2u|\le 2s^\delta\}\\ \le\mathbf{P}\{\tilde{\xi}\in C_s(u,s^\delta)\}\\ \le\mathbf{P}\{1-\cos(\xi_1-\xi_2)\le 2s,\ |\xi_1+\xi_2+\pi-2u|\le 2s^\delta\}\,,\end{aligned}$$

where the addition of angles is understood by modulus 2π. Thus, $\mathbf{P}\{\tilde{\xi}\in C_s(u,s^\delta)\}$ is equivalent as $s\downarrow 0$ to

$$\mathbf{P}\{|\xi_1+\xi_2|\le 2\sqrt{s},\ |\xi_1+\xi_2+\pi-2u|\le 2s^\delta\}\,.$$

Assume that the distribution κ has bounded density f with respect to the length measure on the unit circle. Then the probability above is equivalent to

$$2\sqrt{s}4s^\delta f(u)f(u+\pi)=4s^\gamma(2s^\delta)f(u)f(u+\pi)\,.$$

Thus, (2.4) holds with $\gamma=\frac{1}{2}$ and

$$\sigma_0=4\int_0^{2\pi}f(u)f(u+\pi)du\,.$$

The boundedness of f also implies that (2.5) holds, so that the limit distribution is given by (2.8). In particular if κ is uniform on the circle, then $f(u)=1/(2\pi)$, so that $\sigma_0=4/(2\pi)=2/\pi$, so that

$$\lim_{n\to\infty}\mathbf{P}\{n^4(2-\operatorname{diam}(\Pi_{n\kappa}))\le t\}=1-e^{-\frac{1}{\pi}\sqrt{t}}\,,\quad t\ge 0\,,$$

which also corresponds to the result of Theorem 5.2 for $d=2$.

5.5 Segments and disks in the unit ball

Assume that $L_1, \dots, L_m$ are segments that obtained by intersection the unit ball with m different lines passing through the origin. Assume that $\mathbf{P}\{\xi \in L_i\} = p_i$, $i = 1, \dots, m$, and given $\xi \in L_i$, ξ is distributed according to the length measure on L_i. If $L_i = [-x_i, x_i]$, then let σ be an atomic measure with unit atoms at $\{\pm x_i,\ i = 1, \dots, m\}$. The one-dimensional result for the range of a uniform random sample [6, ???] implies that

$$\lim_{s\downarrow 0} \frac{\mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap C_s(x_i, z_s) \neq \emptyset\}}{\sigma(C(x_i, z_s))} = 1 - \Big(1 + \frac{1}{2} t p_i\Big) e^{-\frac{1}{2} t p_i}\,,$$

where $n = t/s$, i.e. $\gamma = 2$. Theorem 2.9 implies that

$$\lim_{n\to\infty} \mathbf{P}\{n(2 - \operatorname{diam}(\Pi_{n\kappa})) \le t\} = 1 - e^{-\frac{1}{2}t} \prod_{i=1}^{m} \Big(1 + \frac{1}{2} t p_i\Big)\,. \tag{5.4}$$

If $L_1, \dots, L_m$ are obtained as intersections of the unit ball with linear subspaces of possibly different dimensions, then only those with the smallest dimension of these subspaces contribute to the asymptotic distribution of the maximum interpoint distance. If the smallest dimension is at least 2, then σ is non-atomic and Theorem 2.7 is applicable as in the case of uniformly distributed points. Otherwise, we arrive at the formula above for segments.

Assume now that the number of atoms $L_i = [-x_i, x_i]$, $i \ge 1$. Without loss of generality assume that $x_i \to x_0$ as $i \to \infty$ and $x_i \neq x_0$ for all i. Let ν be the measure on $\mathbb{S}^{d-1}$ with atoms $\pm x_i$ with $\nu(\{x_i\}) = \nu(\{-x_i\}) = p_i$, $i \ge 1$. In comparison with the case of a finite number of segments, we need also to find the limit of $\mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap C_s(x_0, z_s) \neq \emptyset\}$ as $s \to 0$. Notice that $\nu(C_s(x_0, z_s)) = q_s \to 0$ as $s \downarrow 0$. Thus, $\mathbf{P}\{\tilde{\Pi}_{n\kappa} \cap C_s(x_0, z_s) \neq \emptyset\}$ is bounded above by the probability that the Poisson point process with the total intensity nq_s on $[-x_0, x_0]$ has diameter that exceeds $2 - s$. Using one-dimensional result, it is easy to see that the corresponding limit is zero if $\gamma = 2$. In order to arrive at a non-trivial limit, we need to set $\gamma > 2$, which is impossible, since the normalisation n^γ is to big for the diameters of the Poisson processes restricted on the individual segments L_i, $i \ge 1$. Therefore, (5.4) holds in this case with the infinite product, i.e. for $m = \infty$.

Foe instance, assume that $p_i = \zeta(2) i^{-2}$, $i \ge 1$, where ζ is the zeta-function. Using a formula for infinite product [10, (89.5.16)] we obtain

$$\lim_{n\to\infty} \mathbf{P}\{n(2 - \operatorname{diam}(\Pi_{n\kappa})) \le t\} = 1 - e^{-\frac{1}{2}t} \frac{1}{\pi} \sqrt{\frac{2\zeta(2)}{t}}\ \sinh \pi \sqrt{\frac{t}{2\zeta(2)}}\,.$$

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